

An Elementary, First Principles Approach to The Indefinite Spin Groups

E. Herzig

Department of Mathematical Sciences
University of Texas at Dallas
Richardson, TX 75080

V. Ramakrishna

Department of Mathematical Sciences
University of Texas at Dallas
Richardson, TX 75080
Corresponding Author

Abstract

In this work we provide an elementary derivation of the indefinite spin groups $Spin^+(p, q)$ in low-dimensions. Our approach relies on the isomorphism $Cl(p+1, q+1) = M(2, Cl(p, q))$, simple properties of Kronecker products, characterization of when an even dimensional real (resp. complex) matrix represents a complex (resp. quaternionic) linear transformation, and basic aspects of the isomorphism $\mathbf{H} \otimes \mathbf{H} = M(4, \mathbf{R})$. Of these the last is arguably the most vital. Among other things it yields a surprisingly ubiquitous role for the matrix $\begin{pmatrix} 0 & i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}$. Our approach has the benefit of identifying these spin groups as explicit groups of matrices within the same collection of matrices which define $Cl(p, q)$. In other words, we do not work in the algebra of matrices that the collection of even vectors in $Cl(p, q)$ is isomorphic to. This is crucial in some applications and also has the advantage of being didactically simple.

1 Introduction

The spin groups play a vital role in theoretical and applied aspects of Clifford Algebras. Though the most popular spin group coverings are those of the rotation groups, the “indefinite” coverings $SL(2, \mathbf{C}) \rightarrow SO^+(3, 1, \mathbf{R})$, $Sp(4, \mathbf{R}) \rightarrow SO^+(3, 2, \mathbf{R})$ and $SU(2, 2) \rightarrow SO^+(4, 2, \mathbf{R})$ are also important in many physical applications, [3, 4, 6, 11].

In the literature the usual derivations of these covering groups usually proceeds by working within the matrix algebra that the collection of even vectors in $Cl(p, q)$ is isomorphic to (see, for instance, the excellent text [15]). Though elegant and important such a derivation is inadequate for some applications and often requires ad hoc constructions.

One application that requires that the spin group be identified as a collection of matrices within the same matrix algebra that the one-vectors of $Cl(p, q)$ live in (thus the matrix algebra $Cl(p, q)$ is itself isomorphic to) is that of computing exponentials of matrices in $so(p, q, \mathbf{R})$. Usage of the covering group provides an algorithm (summarized in Algorithm 1 in the appendix) which renders the problem of exponentiating such matrices tractable, but which requires that the covering group lives in the same matrix algebra as the one-vectors.

Didactically too, it is illuminating to be able to identify the spin group as an explicit group of matrices in the matrix algebra $Cl(p, q)$ is isomorphic to. In other words, it is desirable to be able to explicitly identify (not merely upto isomorphism) the spin group as a collection of matrices in $Cl(p, q)$ emanating from the defining conditions for the spin group [See IV) of Definition (2.1)].

Related to this issue is the fact that the explicit form of the spin group is very much dependent on the choice of one-vectors chosen for $Cl(p, q)$ (and hence on the attendant form of Clifford conjugation, reversion and grade morphisms). Thus a statement such as “ $Spin^+(3, 2)$ is isomorphic to $Sp(4, \mathbf{R})$ ” is

more useful if one knows with respect to which set of one-vectors in $Cl(3, 2)$ (the double ring of $M(4, \mathbf{R})$) this statement pertains to, and thus also how precisely $Sp(4, \mathbf{R})$ sits inside the subcollection of 8×8 real matrices that $Cl(3, 2)$ is isomorphic to.

As a second illustration of this matter, consider the well known fact that $SL(2, \mathbf{C})$ is the double cover of the Lorentz group. This is often demonstrated by first identifying \mathbf{R}^4 with H_2 , the space of 2×2 Hermitian matrices and then observing that i) the quadratic form associating to the generic element, $tI_2 + x\sigma_x + y\sigma_y + z\sigma_z$ of H_2 minus its determinant, viz., $-\det(tI_2 + x\sigma_x + y\sigma_y + z\sigma_z) = x^2 + y^2 + z^2 - t^2$, coincides with the Minkowski metric; and ii) the conjugation action of $SL(2, \mathbf{C})$ on H_2 preserves this quadratic form.

Though undoubtedly elegant and extremely useful in its own right, this derivation begs the question of how precisely $SL(2, \mathbf{C})$ arises from its Clifford theoretic description.

In this work we provide satisfactory resolutions to these questions. The key ingredients in our approach are the following:

- Iterative use of the explicit isomorphism between $Cl(p+1, q+1)$ and $M(2, Cl(p, q))$. This isomorphism provides a ready supply of a basis of 1-vectors for the various Clifford algebras considered here, starting typically with the Pauli matrices (the quintessential anticommuting collection of matrices). The price (reward?) for using these bases of 1-vectors is that the associated version of the spin group is typically not the standard one. In order to reconcile these versions with more familiar versions, the subsequent tools mentioned below play an important role.
- Characterization of when a $2n \times 2n$ real (resp. complex) matrix represents a $n \times n$ complex (resp. quaternionic) linear transformation. This characterization suggests how to modify the basis of 1-vectors provided by the first ingredient above, so as to render the versions of the spin group closer to more familiar realizations. More specifically, this characterization suggests what the desirable form of the grade automorphism ought to be in some cases, and the basis of 1-vectors provided by the first ingredient above is then modified accordingly.
- Basic aspects of the Kronecker products of matrices. The most pertinent use of this is that many of the calculations required in arriving at the descriptions of the spin groups, can be reduced to mere inspection by making use of Kronecker products.
- The most basic aspects of the very useful isomorphism $\mathbf{H} \otimes \mathbf{H} = M(4, \mathbf{R})$. This is, in our view, the important most concomitant. In particular, it provides a basis for $M(4, \mathbf{R})$ consisting of real orthogonal matrices which are either symmetric or antisymmetric. It is the antisymmetric ones which enable seeking alternatives to the standard representative of the symplectic inner product on \mathbf{R}^4 . Furthermore, this basis contains the matrix $\begin{pmatrix} 0 & i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}$, which intervenes in the description of many of the spin groups in this work.

Thus, besides containing an elementary approach to the spin groups, this work also provides novel and sometimes quaint representations of some of the classical groups which arise as the spin groups. In addition, it provides useful interpretations for some of the matrices in the $\mathbf{H} \otimes \mathbf{H}$ basis for $M(4, \mathbf{R})$. In particular, the matrix $M_{1 \otimes k}$ intervenes in an essential and entirely natural fashion in the description of several of the spin groups. In this sense, this work is another contribution to the list of applications of quaternions and Clifford algebras to algorithmic/ computational linear algebra, [1, 7, 5, 12, 13, 16, 17, 2, 18, 8].

Special attention to $Spin^+(3, 2)$ and $Spin^+(4, 2)$, is paid in this work to illustrate the general ideas. These two cases provide the most vivid instances of the ubiquitous role of the matrix $\begin{pmatrix} 0 & i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}$. For the former, we also derive the explicit form of the Lie algebra isomorphism between $spin^+(3, 2)$ and $so(3, 2, \mathbf{R})$. The explicit form of this isomorphism is needed for computing exponentials, for instance. For brevity, derivations for the Lie algebra isomorphism between other $spin^+(p, q)$ and $so(p, q, \mathbf{R})$ have been omitted.

The balance of this manuscript is organized as follows. In the next section all preliminary material required for this paper is collected. We call special attention to Remarks (2.2), (2.3) and (2.4). The third section details the construction of $Spin^+(3, 2)$. The fourth section contains the construction of $Spin^+(4, 2)$. The fifth section develops the remaining spin groups considered in this work - specifically those for which $(p, q) \in \{(2, 1), (2, 2), (3, 1), (4, 1), (1, 5), (3, 3)\}$. The next section offers some conclusions. There is an appendix, which records the algorithm adapted from our earlier work, [8], for the computation of matrix exponentials in $so(p, q, \mathbf{R})$ using the explicit forms of the covering maps developed here.

2 Notation and Preliminary Observations

2.1 Notation

We use the following notation throughout

- 1 \mathbf{H} is the set of quaternions; \mathbf{P} the set of purely imaginary quaternions. Let K be an associative algebra. Then $M(n, K)$ is just the set of $n \times n$ matrices with entries in K . For $X \in M(n, K)$ with $K = \mathbf{C}$ or \mathbf{H} , X^* is the matrix obtained by performing entrywise complex (resp. quaternionic) conjugation first, and then transposition. For $K = \mathbf{C}$, \bar{X} is the matrix obtained by performing entrywise complex conjugation.
- 2 $J_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. Associated to J_{2n} are i) $Sp(2n, K) = \{X \in M(2n, K) : X^T J_{2n} X = J_{2n}\}$. Here K is R or C . $Sp(2n, K)$ is a Lie group; ii) $sp(2n, K) = \{X \in M(2n, K) : X^T J_{2n} = -J_{2n} X\}$. $sp(2n, K)$ is the Lie algebra of $Sp(2n, K)$.
- 3 $\tilde{J}_{2n} = J_2 \oplus J_2 \oplus \dots \oplus J_2$. Thus \tilde{J}_{2n} is the n -fold direct sum of J_2 . \tilde{J}_{2n} , is of course, explicitly permutation similar to J_{2n} , but it is important for our purposes to maintain the distinction. Accordingly $\tilde{Sp}(2n, K) = \{X \in M(2n, K) : \tilde{J}_{2n}^{-1} X^T \tilde{J}_{2n} = J_{2n}\}$. Similarly, $\tilde{sp}(2n, K) = \{X \in M(2n, K) : X^T \tilde{J}_{2n} = -\tilde{J}_{2n} X\}$. Other variants of J_4 are of importance to this paper, and they will be introduced later at appropriate points (see Remark 2.16 below).
- 4 Let $p > 0, q > 0$ and $n = p + q$. Then $I_{p,q} = \text{diag}(I_p - I_q)$. Associated to $I_{p,q}$ is the Lie algebra $so(p, q, R) = \{X \in M(n, R) : X^T I_{p,q} = -I_{p,q} X\}$, sometimes called the Lorenz Lie algebra. We defer definitions of the associated pertinent Lie groups to Section 2. The Lie group $SU(p, q) = \{X \in M(n, C) : X^* I_{p,q} X = I_{p,q}, \det(X) = 1\}$. Its Lie algebra is $su(p, q) = \{X \in M(n, C) : X^* I_{p,q} = -I_{p,q} X, \text{tr}(X) = 0\}$.

- 5 The Pauli Matrices are

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note $i\sigma_y = J_2$.

- 6 The matrix K_{2l} is

$$K_{2l} = \begin{pmatrix} 0_l & I_l \\ I_l & 0_l \end{pmatrix}$$

This matrix enables succinct expressions for Clifford conjugation in some of the Clifford algebras used in this work.

2.2 Reversion, Clifford Conjugation and the Spin Groups

The reader is referred to the excellent texts, [11, 15] for formal definitions of Clifford algebras $Cl(p, q)$. For our purposes it suffices to record the following:

- Definition 2.1 I)** The reversion anti-automorphism on a Clifford algebra, ϕ^{rev} , is the linear map defined by requiring that i) $\phi^{rev}(ab) = \phi^{rev}(b)\phi^{rev}(a)$; ii) $\phi^{rev}(v) = v$, for all 1-vectors v ; and iii) $\phi^{rev}(1) = 1$. For brevity we will write X^{rev} stands for $\phi^{rev}(X)$.
- II)** The Clifford conjugation anti-automorphism on a Clifford algebra, ϕ^{cc} , is the linear map defined by requiring that i) $\phi^{cc}(ab) = \phi^{cc}(b)\phi^{cc}(a)$; ii) $\phi^{cc}(v) = -v$, for all 1-vectors v ; and iii) $\phi^{cc}(1) = 1$. Once again $\phi^{cc}(X)$ is also denoted as X^{cc} .
- III)** The grade automorphism on a Clifford algebra, ϕ^{gr} is $\phi^{rev} \circ \phi^{cc}$. Once again we write X^{gr} for $\phi^{gr}(X)$.
- IV)** $Spin^+(p, q)$ is the collection of elements x in $Cl(p, q)$ satisfying the following requirements: i) $x^{gr} = x$, i.e., x is even; ii) $xx^{cc} = 1$; and iii) For all 1-vectors v in $Cl(p, q)$, xvx^{cc} is also a 1-vector. The last condition, in the presence of the first two conditions, is known to be superfluous for $p + q \leq 5$, [11, 15]. $Spin^+(p, q)$ is a connected Lie group, whose Lie algebra is described in the next item.
- V)** $spin^+(p, q)$ is the Lie algebra of $Spin^+(p, q)$. It is a fact that it also equals the space of bivectors in $Cl(p, q)$, [11, 15]. Thus if $\{X_i, i = 1, \dots, n\}$ is a basis of 1-vectors for $Cl(p, q)$, then the space of bivectors is the real span of $\{X_k X_l, k < l\}$. This space does not, of course, depend on the choice of basis of 1-vectors.

Remark 2.2 The even subalgebra of $Cl(p, q)$ is $Cl(p, q-1)$ [respectively, $Cl(q, p-1)$] if $q \geq 1$ (respectively, if $p \geq 1$). Since the even subalgebras are also Clifford algebras they too are matrix algebras. However, in this work we prefer to view the even subalgebra as a matrix subalgebra of the ambient matrix algebra $Cl(p, q)$. The reason for this is that the analysis of the remaining conditions defining $Spin^+(p, q)$ will then not require any ad hoc constructions.

Remark 2.3 Each $Cl(p, q)$ is a matrix algebra with entries in some associative algebra suitably concocted out of the real numbers, the complex numbers or the quaternions, [11, 15]. On this matrix algebra the explicit matrix forms of reversion, grade and Clifford conjugation depend in an essential fashion on the choice of basis of 1-vectors for $Cl(p, q)$. Once these forms have been obtained, the first two conditions in the definition of $Spin^+(p, q)$ [IV] of Definition (2.1)] will lead to a group of matrices \mathbf{K} in $Cl(p, q)$. The third condition, pertinent only when $p + q \geq 6$, imposes further restrictions on this group. These additional restrictions are best analysed by passing to the Lie algebra $spin^+(p, q)$. In this work these restrictions are analysed in one of two ways: I) Suppose the first two conditions defining $Spin^+(p, q)$ lead to a representation $\Psi(L_1) \subseteq Cl(p, q)$ of a Lie algebra L_1 for the Lie algebra of \mathbf{K} . Then dimension considerations often suggest $\Psi(L_2)$, as a candidate for $spin^+(p, q)$, where L_2 is a codimension one Lie subalgebra of L_1 . One then formally confirms that $\Psi(L_2)$ is indeed $spin^+(p, q)$ by showing that if $X \neq 0$ is in $\Psi(L_1)$ but X does not belong to $\Psi(L_2)$, then X violates the linearization of the third condition in the definition of $Spin^+(p, q)$, i.e., one shows that there is a one vector v such that $Xv - vX$ is not a one-vector. Typically X will be the Ψ image of a multiple of the identity matrix. II) Alternatively one computes explicitly the space of bivectors and attempts to discern a recognizable structure of a Lie algebra of matrices on it.

Remark 2.4 The even subalgebra of $Cl(p, q)$ is evidently important in the description of the spin group. Therefore, to the extent possible, it is vital to be able to identify it as a “recognizable” matrix subalgebra of the matrix algebra $Cl(p, q)$ is isomorphic to. Equivalently it is useful to work with a convenient form for the grade isomorphism. However, this depends on the choice of basis of one-vectors. In this work, starting with a basis of 1-vectors, the grade isomorphism will always take the form $X \rightarrow M^{-1}\psi(X)M$, where

M is some unitary matrix and $\psi(X)$ is either X or \bar{X} (where \bar{X} is entrywise complex or quaternionic conjugation). Suppose now that it is preferable for the grade isomorphism to read as $X \rightarrow N^{-1}\psi(X)N$, with N a matrix explicitly conjugate to M , via $N = P^{-1}MP$. This is then achieved by changing the original basis of one-vectors X_i to a new basis of 1-vectors $Y_i = P^{-1}X_iP$. Furthermore, suppose Clifford conjugation with respect to the $\{X_i\}$ reads as $X \rightarrow C^{-1}\phi(X)C$, where $\phi(X)$ is either X^T or X^* (where $*$ is Hermitian complex or quaternionic conjugation) and C some unitary matrix, as will be the case throughout this work. Then Clifford conjugation, with respect to the new basis $\{Y_i\}$, is given by $X \rightarrow D^{-1}\phi(X)D$, with $D = P^{-1}CP$. It is emphasized that our techniques render the finding of such a conjugation P fully constructive - eigencomputations going beyond 2×2 matrices are never invoked.

Definition 2.5 The Lorenz group, $SO^+(p, q, \mathbf{R})$ is the connected component of the identity of the group $SO(p, q, \mathbf{R}) = \{X : X^T I_{p,q} X = I_{p,q}, \det(X) = 1\}$. Its Lie algebra (as well as the Lie algebra of $SO(p, q, \mathbf{R})$) is $so(p, q, \mathbf{R})$. It is known that the map which assigns to any element $g \in Spin^+(p, q)$ the matrix of the linear map $v \rightarrow gvg^{cc}$, with v a 1-vector in $Cl(p, q)$ is a $2 : 1$ group homomorphism from $Spin^+(p, q)$ to $SO^+(p, q, \mathbf{R})$. The map obtained by linearizing this map, viz., the matrix of $v \rightarrow Xv - vX$ (where v is a 1-vector and X an element of $spin^+(p, q)$), is a Lie algebra isomorphism from $spin^+(p, q)$ to $so(p, q, \mathbf{R})$.

2.3 An Iterative Construction

Here we will outline an iterative construction of 1-vectors for certain Clifford Algebras, given a choice of one vectors for another Clifford Algebra, [11, 15]. Throughout this work this construction, together with the contents of Remarks (2.6) below, is collectively referred to as **IC**.

IC $Cl(p+1, q+1)$ as $M(2, Cl(p, q))$, where $M(2, \mathfrak{A})$ stands for the set of 2×2 matrices with entries in an associative algebra \mathfrak{A} : Suppose $\{e_1, \dots, e_p, f_1, \dots, f_q\}$ is a basis of 1-vectors for $Cl(p, q)$. So, in particular, $e_k^2 = +1$, $k = 1, \dots, p$ and $f_l^2 = -1$, $l = 1, \dots, q$. Then a basis of 1-vectors for $Cl(p+1, q+1)$ is given by the following collection of elements in $M(2, Cl(p, q))$:

$$\begin{pmatrix} e_k & 0 \\ 0 & -e_k \end{pmatrix}, k = 1, \dots, p; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} f_l & 0 \\ 0 & -f_l \end{pmatrix}, l = 1, \dots, q; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The 1 and the 0 in the matrices above are the identity and zero elements of $Cl(p, q)$ respectively.

Remark 2.6 If Clifford conjugation and reversion have been identified on $Cl(p, q)$ with respect to some basis of 1-vectors, then there are explicit expressions for Clifford conjugation and reversion on $Cl(p+1, q+1)$ with respect to the basis of 1-vectors described in iterative construction **IC1** above.

Specifically if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then we have

$$X^{CC} = \begin{pmatrix} D^{rev} & -B^{rev} \\ -C^{rev} & A^{rev} \end{pmatrix}$$

while reversion is

$$X^{rev} = \begin{pmatrix} D^{cc} & B^{cc} \\ C^{cc} & A^{cc} \end{pmatrix}$$

This is immediate from the definitions of reversion and Clifford conjugation.

Note that if elements of $Cl(p, q)$ have been identified with $l \times l$ matrices, then

$$X^{cc} = J_{2l}^{-1} \left[\begin{pmatrix} A^{rev} & B^{rev} \\ C^{rev} & D^{rev} \end{pmatrix} \right]^{BT} J_{2l}$$

and that

$$X^{rev} = K_{2l}^{-1} \left[\begin{pmatrix} A^{cc} & B^{cc} \\ C^{cc} & D^{cc} \end{pmatrix} \right]^{BT} K_{2l}$$

where K_{2l} is the matrix at the end of Section 2.1, and if $X = \begin{pmatrix} Y & Z \\ U & V \end{pmatrix}$ is a 2×2 block matrix, then

$$X^{BT} = \begin{pmatrix} Y & U \\ Z & V \end{pmatrix}$$

2.4 $\theta_{\mathbb{C}}$ and $\theta_{\mathbb{H}}$ matrices:

Some of the material here is to be found in [9], for instance.

Definition 2.7 Given a matrix $M \in M(n, \mathbb{C})$, define a matrix $\theta_{\mathbb{C}, \mathbb{I}}(M) \in M(2n, \mathbb{R})$ by first setting $\theta_{\mathbb{C}, \mathbb{I}}(z) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ for a complex scalar $z = x + iy$. We then define $\theta_{\mathbb{C}}(M) = (\theta_{\mathbb{C}}(m_{ij}))$, i.e., $\theta_{\mathbb{C}}(M)$ is a $n \times n$ block matrix, with the (i, j) th block equal to the 2×2 real matrix $\theta_{\mathbb{C}}(m_{ij})$.

Remark 2.8 Properties of $\theta_{\mathbb{C}, \mathbb{I}}$: Some useful properties of the map $\theta_{\mathbb{C}, \mathbb{I}}$ now follow:

- i) $\theta_{\mathbb{C}, \mathbb{I}}$ is an \mathbf{R} -linear map.
- ii) $\theta_{\mathbb{C}, \mathbb{I}}$ is an associative algebra isomorphism onto its image from $M(n, \mathbf{C})$ to $M(2n, \mathbf{R})$.
- iii) $\theta_{\mathbb{C}, \mathbb{I}}(M^*) = [\theta_{\mathbb{C}, \mathbb{I}}(M)]^T$
- iv) $\theta_{\mathbb{C}, \mathbb{I}}(I_n) = I_{2n}$
- v) A useful property is the following: $X \in M(2n, \mathbf{R})$ is in the image of $\theta_{\mathbb{C}, \mathbb{I}}$ iff $X^T = \tilde{J}_{2n}^{-1} X^T \tilde{J}_{2n}$ iff $X \tilde{J}_{2n} = X \tilde{J}_{2n}$.

Remark 2.9 We call an $X \in \text{Im}(\theta_{\mathbb{C}, \mathbb{I}})$ a $\theta_{\mathbb{C}, \mathbb{I}}$ matrix.

Remark 2.10 An closely related alternative way to associate real matrices to matrices in $M(n, \mathbf{C})$ is as follows. Write $X \in M(n, \mathbf{C})$ as $X = Y + iZ$ with Y, Z both real matrices. The define $\Theta_{\mathbf{C}, \mathbf{II}}(Z) = \begin{pmatrix} X & -Y \\ Y & Z \end{pmatrix}$. All contents of Remark (2.8) apply verbatim except v) which is now replaced by “ $X \in M(2n, \mathbf{R})$ is in the image $\Theta_{\mathbf{C}, \mathbf{II}}$ iff $X J_{2n} = J_{2n} X$ ”.

Next, to a matrix with quaternion entries will be associated a complex matrix. First, if $q \in \mathbb{H}$ is a quaternion, it can be written uniquely in the form $q = z + wj$, for some $z, w \in \mathbb{C}$. Note that $j\eta = \bar{\eta}j$, for any $\eta \in \mathbb{C}$. With this at hand, the following construction associating complex matrices to matrices with quaternionic entries (see [9] for instance) is useful:

Definition 2.11 Let $X \in M(n, \mathbb{H})$. By writing each entry x_{pq} of X as

$$x_{pq} = z_{pq} + w_{pq}j, \quad z_{pq}, w_{pq} \in \mathbb{C}$$

we can write X uniquely as $X = Z + Wj$ with $Z, W \in M(n, \mathbb{C})$. Associate to X the following matrix $\theta_{\mathbb{H}}(X) \in M(2n, \mathbb{C})$:

$$\theta_{\mathbb{H}}(X) = \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix}$$

Next some useful properties of the map $\theta_{\mathbb{H}} : M(n, \mathbb{H}) \rightarrow M(2n, \mathbb{C})$ are collected.

Remark 2.12 Properties of $\theta_{\mathbb{H}}$:

- i) $\theta_{\mathbb{H}}$ is an \mathbf{R} linear map.

ii) $\theta_{\mathbb{H}}(XY) = \theta_{\mathbb{H}}(X)\theta_{\mathbb{H}}(Y)$

iii) $\theta_{\mathbb{H}}(X^*) = [\theta_{\mathbb{H}}(X)]^*$. Here the $*$ on the left is quaternionic Hermitian conjugation, while that on the right is complex Hermitian conjugation.

iv) $\theta_{\mathbb{H}}(I_n) = I_{2n}$

v) A less known property is the following: $\Lambda \in M(2n, \mathbf{C})$ is in the image of $\theta_{\mathbb{H}}$ iff $\Lambda^* = J_{2n}^{-1} X^T J_{2n}$.

Remark 2.13 We call an $\Lambda \in \text{Im}(\theta_{\mathbf{H}})$, a $\theta_{\mathbf{H}}$ matrix. In [9] such matrices are called matrices of the quaternion type. But we eschew this nomenclature.

Definition 2.14 The determinant of a matrix $X \in M(n, \mathbf{H})$ is defined to be the determinant of the complex matrix $\Theta_{\mathbf{H}}(X)$. $SL(n, \mathbf{H})$ is the group of $n \times n$ quaternionic matrices with unit determinant. Its Lie algebra, $sl(n, \mathbf{H})$ is the subset of $M(n, \mathbf{H})$ satisfying $\text{Re}(\text{Tr } X) = 0$.

Remark 2.15 Though the contents of this remark seem unrelated to the rest of this subsection, they play a similar utilitarian role in this work. Specifically, a matrix with entries in \mathbf{C} commutes with a diagonal matrix whose equal entries appear contiguously, if and only if the matrix is block-diagonal, [9]. In this work this diagonal matrix will be $I_{r,s}$ (in which case this result applies even for matrices with entries in \mathbf{H}). This observation will be used, when applicable, to change the basis of 1-vectors via permutation similarities so as to render the even subalgebra equal to an algebra of block-diagonal matrices.

2.5 $\mathbf{H} \otimes \mathbf{H}$ and $M(4, \mathbf{R})$

The algebra isomorphism between $\mathbb{H} \otimes \mathbb{H}$ and $M(4, \mathbf{R})$ (also denoted by $gl(4, \mathbf{R})$) may be summarized as follows:

- Associate to each product tensor $p \otimes q \in \mathbb{H} \otimes \mathbb{H}$, the matrix, $M_{p \otimes q}$, of the map which sends $x \in \mathbb{H}$ to $px\bar{q}$, identifying \mathbb{R}^4 with \mathbb{H} via the basis $\{1, i, j, k\}$. Here, $\bar{q} = q_0 - q_1i - q_2j - q_3k$
- Extend this to the full tensor product by linearity. This yields an associative algebra isomorphism between $\mathbb{H} \otimes \mathbb{H}$ and $M(4, \mathbf{R})$. Furthermore, a basis for $gl(4, \mathbf{R})$ is provided by the sixteen matrices $M_{e_x \otimes e_y}$ as e_x, e_y run through $1, i, j, k$.
- We define conjugation on $\mathbb{H} \otimes \mathbb{H}$ by setting $p \bar{\otimes} q = \bar{p} \otimes \bar{q}$ and then extending by linearity. Conjugation in $\mathbb{H} \otimes \mathbb{H}$ corresponds to matrix transposition, i.e., $M_{\bar{p} \otimes \bar{q}} = (M_{p \otimes q})^T$. A consequence of this is that any matrix of the form $M_{1 \otimes p}$ or $M_{q \otimes 1}$, with $p, q \in \mathbb{P}$ is a real antisymmetric matrix. Similarly, the most general special orthogonal matrix in $M(4, \mathbf{R})$ admits an expression of the form $M_{p \otimes q}$, with p and q both unit quaternions.

Remark 2.16 Some important matrices from this basis for $M(4, \mathbf{R})$ provided by $\mathbb{H} \otimes \mathbb{H}$ are:

- $M_{1 \otimes j}$ is precisely J_4 .
- The matrix $M_{1 \otimes k}$, which we denote by \check{J}_4 . Note also that $M_{1 \otimes k}$ is a Kronecker product, viz., $\sigma_x \otimes i\sigma_y$.
- $M_{i \otimes 1} = -\tilde{J}_4$.

2.6 Kronecker Products

The Kronecker product $A \otimes B$, [9] has the following properties which will be used throughout this work.

- **KP1** $(A \otimes B)(C \otimes D) = AC \otimes BD$. $(A \otimes B)^T = A^T \otimes B^T$.
- **KP2** A special case of **KP1**, worth recording separately, is the following. If (λ, v) and (μ, w) are eigenpairs of $A_{n \times n}$ and $B_{m \times m}$ respectively, then $(\lambda\mu, v \otimes w)$ is an eigenpair for $A \otimes B$. This observation will be used in Section 4 to produce a certain desired conjugation by mere inspection.
- **KP3** If A and B are square then $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$.

3 $\text{Spin}^+(3, 2)$

It is convenient to begin with $\{\sigma_z\}$ as a basis of 1-vectors for $Cl(1, 0)$. Quite clearly reversion with respect to this basis is $X \rightarrow X^T$. Similarly, Clifford conjugation is $X \rightarrow J_2^T X^T J_2$. Applying **IC** to this basis yields the following basis for $Cl(2, 1)$:

$$\{\sigma_z \otimes \sigma_z, \sigma_x \otimes I_2, i\sigma_y \otimes I_2\} \quad (1)$$

Hence by **IC** reversion on $Cl(2, 1)$, with respect to this basis is

$$X \rightarrow R_{2,1}^T X^T R_{2,1}; R_{2,1} = \begin{pmatrix} 0_2 & J_2 \\ J_2 & 0_2 \end{pmatrix} = M_{1 \otimes k} \quad (2)$$

Clifford conjugation on $Cl(2, 1)$, with respect to this basis, is

$$X \rightarrow C_{2,1}^T X^T C_{2,1}, C_{2,1} = J_4 \quad (3)$$

Applying **IC** again produces the following basis of 1-vectors for $Cl(3, 2)$:

$$\{\sigma_z \otimes \sigma_z \otimes \sigma_z, \sigma_z \otimes \sigma_x \otimes I_2, \sigma_x \otimes I_2 \otimes I_2, \sigma_z \otimes i\sigma_y \otimes I_2, i\sigma_y \otimes I_2 \otimes I_z\} \quad (4)$$

As is well known $Cl(3, 2)$ is the double ring of $M(4, \mathbf{R})$. Specifically, the last basis renders $Cl(3, 2)$ equal to the subalgebra of $M(8, \mathbf{R})$, consisting of matrices which when written as 4×4 block matrices, have each block equal to a diagonal 2×2 matrix.

Reversion with respect to this last basis for $Cl(3, 2)$ is thus given by

$$X \rightarrow R_{3,2}^T X^T R_{3,2}; R_{3,2} = \begin{pmatrix} 0_4 & J_4 \\ J_4 & 0_4 \end{pmatrix} \quad (5)$$

Clifford conjugation on $Cl(3, 2)$ is provided is expressible as

$$X \rightarrow C_{3,2}^T X^T C_{3,2}; C_{3,2} = \begin{pmatrix} 0_4 & M_{1 \otimes k} \\ -M_{1 \otimes k} & 0_4 \end{pmatrix} \quad (6)$$

Hence the grade involution on $Cl(3, 2)$ is given by

$$X \rightarrow G_{3,2}^T X G_{3,2}, G_{3,2} = \begin{pmatrix} -M_{1 \otimes i} & 0_4 \\ 0_4 & M_{1 \otimes i} \end{pmatrix} \quad (7)$$

Hence $\text{Spin}_+(3, 2)$ equals the subalgebra of matrices $X \in M(8, \mathbf{R})$ characterized by the following conditions:

- $X = (X_{ij}), i, j = 1, \dots, 4$, with each X_{ij} a 2×2 diagonal matrix.
- $G_{3,2}$ commutes with X .

- $X^T C_{3,2} X = C_{3,2}$

The first two conditions, as a calculation shows, impose the following restrictions on the blocks of X :

- $X_{11}, X_{14}, X_{22}, X_{23}, X_{32}, X_{33}, X_{41}$ and X_{44} are 2×2 skew-Hamiltonian matrices, and thus 2×2 constant multiples of I_2 .
- The remaining X_{ij} are 2×2 , diagonal, Hamiltonian matrices. Since 2×2 Hamiltonian matrices are precisely the traceless 2×2 matrices, this forces each of these X_{ij} to be a constant multiple of σ_z .

Hence X being an even element of $Cl(3, 2)$ is equivalent to asserting that it has the following form

$$X = \begin{pmatrix} a & 0 & b & 0 & c & 0 & d & 0 \\ 0 & a & 0 & -b & 0 & -c & 0 & d \\ e & 0 & f & 0 & g & 0 & h & 0 \\ 0 & -e & 0 & f & 0 & g & 0 & -h \\ i & 0 & j & 0 & k & 0 & l & 0 \\ 0 & -i & 0 & j & 0 & k & 0 & -l \\ m & 0 & n & 0 & p & 0 & q & 0 \\ 0 & m & 0 & -n & 0 & -p & 0 & q \end{pmatrix}$$

Let us now examine the condition $C_{3,2}^T X^T C_{3,2} X = I_8$. Some calculations which systematically use the Hamiltonian/skew-Hamiltonian structures of the blocks X_{ij} of X reveals the following conditions to be equivalent to $C_{3,2}^T X^T C_{3,2} X = I_8$:

1. The conditions corresponding to the equality of the (1,1) blocks and also the (4,4) blocks are identical to $aq - dm + ih - el = 1$.
2. The conditions corresponding to the equality of the (1,2) blocks and also the (3,4) blocks are identical to $qb - fl + hj - dn = 0$.
3. The equality of the (1,3) blocks and also the (2,4) blocks yield the condition $kh - dp + qc - lg = 0$.
4. The equality of the (2,1) blocks and also the (4,3) blocks yield the condition $cm - gi + ke - ap = 0$.
5. The equality of the (2,2) blocks and similarly that of the (3,3) blocks impose the condition $cn - gj + kf - bp = 1$.
6. The equality of the (3,1) blocks and similarly that of the (4,2) blocks impose the condition $fi - bm + an - je = 0$.
7. The equality of the remaining blocks holds automatically and thus imposes no further restriction on the entries of X .

Let us extract information from these conditions as follows. Define a 4×4 matrix associated to an even vector X as follows:

$$Z = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{pmatrix} \quad (8)$$

Note the relation between Z and X can be rewritten via a map Λ as

$$X = \Lambda(Z) = \begin{pmatrix} aI_2 & b\sigma_z & c\sigma_z & dI_2 \\ e\sigma_z & fI_2 & gI_2 & h\sigma_z \\ i\sigma_z & jI_2 & kI_2 & l\sigma_z \\ mI_2 & n\sigma_z & p\sigma_z & qI_2 \end{pmatrix} \quad (9)$$

Λ is an algebra isomorphism of $M(4, \mathbf{R})$ onto its image in $M(8, \mathbf{R})$ which furthermore satisfies $\Lambda(Z^T) = [\Lambda(Z)]^T$.

Let us also collect the six distinct relations characterizing the $X^{cc}X = I$ condition again for ease of reference:

$$\begin{aligned} aq - dm + ih - el &= 1 \\ bq - fl + hj - dn &= 0 \\ kh - dp + qc - gl &= 0 \\ cm - gi + ke - ap &= 0 \\ cn - gj + kf - bp &= 1 \\ fi - bm + an - je &= 0 \end{aligned} \tag{10}$$

Then, after some experimentation and invocation of the the basis of $M(4, \mathbf{R})$ provided by its isomorphism with $\mathbf{H} \otimes \mathbf{H}$, the conditions in Equation (10) are seen to be equivalent to Z belonging to a nonstandard representation of $Sp(4, \mathbf{R})$. Specifically, defining

$$\check{J}_4 = M_{1 \otimes k} = \begin{pmatrix} 0_2 & J_2 \\ J_2 & 0_2 \end{pmatrix}$$

the conditions in Equation (10) are equivalent to

$$Z^T \check{J}_4 Z = \check{J}_4 \tag{11}$$

To see this it is better to write Z in Equation (8) as a 2×2 block matrix, viz., $Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ [so that $A = \begin{pmatrix} a & b \\ e & f \end{pmatrix}$; $B = \begin{pmatrix} c & d \\ g & h \end{pmatrix}$; $C = \begin{pmatrix} i & j \\ m & n \end{pmatrix}$; and $D = \begin{pmatrix} k & l \\ p & q \end{pmatrix}$]. Then the condition $Z^T \check{J}_4 Z = \check{J}_4$ is equivalent to:

- $C^T J_2 A + A^T J_2 C = 0$, i.e., $C^T J_2 A$ is symmetric.
- $C^T J_2 B + A^T J_2 D = J_2$.
- $D^T J_2 A + B^T J_2 C = J_2$. Note this condition is identical to the previous one, since J_2 is antisymmetric.
- $D^T J_2 B + B^T J_2 D = O_2$, i.e., $D^T J_2 B$ is symmetric.

Now the $(1, 2)$ entry and the $(2, 1)$ entry of $C^T J_2 A$ are $-bm + if$ and $-an + je$ respectively. Thus $C^T J_2 A$ being symmetric is precisely the sixth of the conditions in Equation (10). Next, the $(1, 2)$ entry and the $(2, 1)$ entry of $D^T J_2 B$ are $-pd + kh$ and $-qc + lg$ respectively. Thus $D^T J_2 B$ being symmetric is precisely the third of the conditions in Equation (10).

Finally, a direct calculation shows that $C^T J_2 B + A^T J_2 D = J_2$ is equivalent to the first, second, fourth and the fifth conditions in in Equation (10).

Thus, indeed $Spin^+(3, 2)$, with respect to the basis of 1-vectors in Equation (4), is isomorphic to a nonstandard representation of $Sp(4, R)$. Thus we are lead to the following

Theorem 3.1 Consider the basis of 1-vectors for $Cl(3, 2)$, given by $\{X_1, \dots, X_5\} = \{\sigma_z \otimes \sigma_z \otimes \sigma_z, \sigma_z \otimes \sigma_x \otimes I_2, \sigma_x \otimes I_2 \otimes I_2, \sigma_z \otimes i\sigma_y \otimes I_2, i\sigma_y \otimes I_2 \otimes I_z\}$. With respect to this basis,

- i) Clifford Conjugation is given by Equation (6).
- ii) Reversion is given by Equation (5).
- iii) Grade involution is given by Equation (7).

- $Spin^+(3, 2)$ is given by the subset of $M(8, R)$, admitting a representation of the form

$$\Lambda(X) = \begin{pmatrix} aI_2 & b\sigma_z & c\sigma_z & dI_2 \\ e\sigma_z & fI_2 & gI_2 & h\sigma_z \\ i\sigma_z & jI_2 & kI_2 & l\sigma_z \\ mI_2 & n\sigma_z & p\sigma_z & qI_2 \end{pmatrix}$$

where the real 4×4 matrix

$$X = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{pmatrix}$$

satisfies $X^T M_{1 \otimes k} X = M_{1 \otimes k}$. The collection of such 4×4 matrices, denoted $\check{Sp}(4, \mathbf{R})$, is isomorphic to (the standard representation of) $Sp(4, \mathbf{R})$, via an explicit conjugation.

Let us now use this to compute explicitly the isomorphism between $\check{sp}(4, \mathbf{R}) = \{Y \in M(4, \mathbf{R}) : Y^T M_{1 \otimes k} = -M_{1 \otimes k} Y\}$ and $so(3, 2, \mathbf{R})$. To this end we proceed in 3 steps:

- Use as the basis of $\check{sp}(4, R)$ the collection of matrices $Z_i, i = 1, \dots, 10$, where each Z_i equals $M_{1 \otimes k} W_i$, with $W_i, i = 1, \dots, 10$ ranging over the set $\{I_4\} \cup \{M_{a \otimes b}\}$, where a, b range over the set $\{i, j, k\} \subseteq H$. Note this latter union is a basis of all the symmetric real 4×4 matrix.

- Next embed each $Z_i = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{pmatrix}$ as the 8×8 matrix

$$\Lambda(Z_i) = \begin{pmatrix} aI_2 & b\sigma_z & c\sigma_z & dI_2 \\ e\sigma_z & fI_2 & gI_2 & h\sigma_z \\ i\sigma_z & jI_2 & kI_2 & l\sigma_z \\ mI_2 & n\sigma_z & p\sigma_z & qI_2 \end{pmatrix} \quad (12)$$

- Then compute the 5×5 matrices $M_i, i = 1, \dots, 10$, of the linear maps, $X \rightarrow \theta(Z_i)X - X\theta(Z_i)$ on the space of 1-vectors for $Cl(3, 2)$, with respect to the basis $\{X_1, \dots, X_5\} = \{\sigma_z \otimes \sigma_z \otimes \sigma_z, \sigma_z \otimes \sigma_x \otimes I_2, \sigma_x \otimes I_2 \otimes I_2, \sigma_z \otimes i\sigma_y \otimes I_2, i\sigma_y \otimes I_2 \otimes I_z\}$ for the 1-vectors of $Cl(3, 2)$. We note here that this computation is greatly facilitated by the fact that each of the $\theta(Z_i)$ also admits an expression as a triple Kronecker product of (essentially) the Pauli matrices. *This again attests to the remarkable utility of the $H \otimes H$ basis for $M(4, \mathbf{R})$.*

This leads to the following theorem

Theorem 3.2 The Lie algebra isomorphism $\Psi_{3,2} : \check{sp}(4, \mathbf{R}) \rightarrow so(3, 2, \mathbf{R})$ is given by the following table, wherein the second column forms a basis for $\check{sp}(4, \mathbf{R})$ and the last column forms a basis for $so(3, 2, R)$:

W_i	Z_i	$\Lambda(Z_i)$	M_i
I_4	$M_{1 \otimes k}$	$\sigma_x \otimes (i\sigma_y) \otimes I_2$	$2(e_5 e_4^T - e_4 e_5^T)$
$M_{i \otimes i}$	$M_{i \otimes j}$	$-i\sigma_y \otimes i\sigma_y \otimes I_2$	$-2(e_4 e_3^T + e_3 e_4^T)$
$M_{j \otimes j}$	$-M_{j \otimes i}$	$\sigma_x \otimes \sigma_x \otimes I_2$	$2(e_5 e_2^T + e_2 e_5^T)$
$M_{k \otimes k}$	$-M_{k \otimes 1}$	$i\sigma_y \otimes \sigma_x \otimes I_2$	$2(e_2 e_3^T - e_3 e_2^T)$
$M_{i \otimes j}$	$-M_{i \otimes i}$	$-\sigma_z \otimes I_2 \otimes I_2$	$-2(e_5 e_3^T + e_3 e_5^T)$
$M_{i \otimes k}$	$-M_{i \otimes 1}$	$I_2 \otimes i\sigma_y \otimes \sigma_z$	$2(e_1 e_2^T - e_2 e_1^T)$
$M_{j \otimes i}$	$-M_{j \otimes j}$	$I_2 \otimes \sigma_z \otimes I_2$	$2(e_4 e_2^T - e_2 e_4^T)$
$M_{j \otimes k}$	$-M_{j \otimes 1}$	$i\sigma_y \otimes \sigma_z \otimes \sigma_z$	$2(e_1 e_3^T - e_3 e_1^T)$
$M_{k \otimes i}$	$M_{k \otimes j}$	$I_2 \otimes \sigma_x \otimes \sigma_z$	$-2(e_4 e_1^T + e_4 e_1^T)$
$M_{k \otimes j}$	$-M_{k \otimes i}$	$\sigma_x \otimes \sigma_z \otimes \sigma_z$	$2(e_5 e_1^T - e_1 e_5^T)$

◇

Proof: The proof is a detailed calculation following the strategy outlined before the statement of the theorem. We will provide the details for the first row of the table: $W_1 = I_4$, $Z_1 = M_{1 \otimes k}$, $\Lambda(Z_1) = \sigma_x \otimes (i\sigma_y) \otimes I_2$. Then $\Lambda(Z_1)X_i - X_i\Lambda(Z_1) = 0$ for all $i \in \{1, 2, 3, 4, 5\}$ except $i = 4, 5$ when we, in fact, have $\Lambda(Z_1)X_4 - X_4\Lambda(Z_1) = 2X_5$ and $\Lambda(Z_1)X_5 - X_5\Lambda(Z_1) = -2X_4$. Thus $M_1 = 2(e_5e_4^T - e_4e_5^T)$. ◇

Remark 3.3 Exponentiating Matrices in $so(3, 2, \mathbf{R})$: Theorems (3.1) and (3.2), together with the Algorithm in the appendix provide a constructive procedure, requiring no eigencomputations, for exponentiating a matrix $X \in so(3, 2, \mathbf{R})$. We first compute the element $Z = \Psi_{3,2}^{-1}(X) \in \check{sp}(4, \mathbf{R})$ using the II and IV column of the table in Theorem (3.2). Suppose the matrix $e^Z \in \check{Sp}(4, \mathbf{R})$ has been computed. Then the matrix $\Lambda(e^Z) \in Cl(3, 2) \subseteq M(8, \mathbf{R})$ is found (this being a matter of mere inspection). Then one computes

$$M_i = [\Lambda(e^Z)]X_i[C_{3,2}^T \Lambda(e^{Z^T})C_{3,2}], i = 1, \dots, 5$$

with X_i as in Theorem (3.1) and $C_{3,2}$ as in Equation (6). Note use has been made of the fact that the embedding Λ in Equation (9) satisfies $[\Lambda(X)]^T = \Lambda(X^T)$. The computation of M_i is straightforward in view of the form of the 8×8 matrix $\Theta(e^Z)$. Then $M_i = \sum_{j=1}^5 g_{ji}X_i$, where the real numbers g_{ji} can be computed by using the trace inner product (this computation being facilitated by the fact that the one vectors X_i are triple Kronecker products). The matrix $(g_{ij}), i, j = 1, \dots, 5$ then is precisely e^X . So everything reduces to the computation of e^Z for $Z \in \check{sp}(4, \mathbf{R})$. But this can also be done explicitly in closed form. Indeed, if q is any unit quaternion such that $\bar{q}kq = j$, then the matrix $M_{1 \otimes q}^T Z M_{1 \otimes q}$ is a 4×4 Hamiltonian matrix and hence their minimal polynomials and exponentials can be computed, without eigencomputations in closed form. Hence so too can e^Z . Passage to $sp(4, \mathbf{R})$ can be eschewed by directly working with Z and using the methods of of [18] if one prefers.

4 $Spin^+(4, 2)$

One begins with $Cl(2, 0)$, represented by the following basis of 1-vectors:

$$\{\sigma_x, \sigma_z\} \tag{13}$$

In view of the fact that σ_x, σ_z are symmetric it follows that reversion on $Cl(2, 0)$ is

$$X \rightarrow X^T \tag{14}$$

Similarly since each of σ_x, σ_z is similar to minus itself via $J_2 = i\sigma_y$ it follows that that, with respect to this basis of 1-vectors, Clifford conjugation on $Cl(2, 0)$ is the map

$$X \rightarrow J_2^T X^T J_2 \tag{15}$$

Now we apply **IC** to the foregoing to produce the following basis of 1-vectors for $Cl(3, 1)$

$$\{\sigma_z \otimes \sigma_x, \sigma_z \otimes \sigma_z, \sigma_x \otimes I_2, i\sigma_y \otimes I_2\} \tag{16}$$

IC together with some block matrix calculations shows that reversion on $Cl(3, 1)$, with respect to the basis in Equation (16), is given by

$$X \rightarrow P_4^T X^T P_4$$

where $P_4 = M_{1 \otimes k}$ again.

Similarly, Clifford conjugation on $Cl(3, 1)$ is $X \rightarrow J_4^T X^T J_4$. Thus the grade involution on $Cl(3, 1)$ is therefore $X \rightarrow (-M_{1 \otimes i})^T X (-M_{1 \otimes i})$.

Let us now apply **IC** again to the above basis of 1-vectors and forms of reversion and Clifford conjugation on $Cl(3, 1)$ to produce the following basis of 1-vectors for $Cl(4, 2)$:

$$\{\sigma_z \otimes \sigma_z \otimes \sigma_x, \sigma_z \otimes \sigma_z \otimes \sigma_z, \sigma_z \otimes \sigma_x \otimes I_2, \sigma_x \otimes I_2 \otimes I_2 = K_8, \sigma_z \otimes i\sigma_y \otimes I_2, i\sigma_y \otimes I_2 \otimes I_2 = J_8\} \quad (17)$$

Then reversion on $Cl(4, 2)$ is, with respect to this basis of 1-vectors, given by

$$X \rightarrow P_8^T X^T P_8 \quad (18)$$

where P_8 is the product $\begin{pmatrix} J_4 & 0_4 \\ 0_4 & J_4 \end{pmatrix} K_8 = \sigma_x \otimes i\sigma_y \otimes I_2$.

Similarly, Clifford conjugation $Cl(4, 2)$ is, with respect to this basis of 1-vectors, given by

$$X \rightarrow M_8^T X^T M_8, M_8 = i\sigma_y \otimes \sigma_x \otimes i\sigma_y \quad (19)$$

Therefore, the grade involution $Cl(4, 2)$ is, with respect to the basis in Equation (17),

$$X \rightarrow G_8^T X G_8, G_8 = \text{diag}(-J_2, J_2, J_2, -J_2) \quad (20)$$

where, in arriving at the form G_8 , systematic use of Equation **KP1** of Section 2.6 was made.

In view of the map $\theta_{C, I}$, we would like the grade involution to be $X \rightarrow G_{4,2}^T X G_{4,2}$, with $G_{4,2} = \tilde{J}_8$.

Since $\sigma_z^{-1} J_2 \sigma_z = \sigma_z J_2 \sigma_z = -J_2$, it is seen that G_8 and $G_{4,2}$ are explicitly conjugate

$$S^{-1} G_8 S = G_{4,2}$$

with

$$S = S^{-1} = S^T = \text{diag}[\sigma_z, I_2, I_2, \sigma_z]$$

Accordingly we also change the basis of 1-vectors for $Cl(4, 2)$. If we let the basis members in Equation (17) by $\{X_i, i = 1, \dots, 6\}$, then we define a second basis of 1-vectors for $Cl(4, 2)$ by $Y_i = S^{-1} X_i S, i = 1, \dots, 6$.

This leads to the following basis:

$$\{-\sigma_z \otimes \sigma_z \otimes \sigma_x, \sigma_z \otimes \sigma_z \otimes \sigma_z, \sigma_z \otimes \sigma_x \otimes \sigma_z, \sigma_x \otimes I_2 \otimes \sigma_z, \sigma_z \otimes i\sigma_y \otimes \sigma_z, i\sigma_y \otimes \sigma_z\} \quad (21)$$

Then, in view of Remark (2.4), the grade involution on $Cl(4, 2)$ is indeed given by conjugation with respect to $G_{4,2}$ as desired.

Defining, $C_{4,2} = S^T M_8 S$, we find by Remark (2.4) that Clifford conjugation is, with respect to the basis in Equation (21), given by $Y \rightarrow C_{4,2}^T Y^T C_{4,2}, \forall Y \in Cl(4, 2)$.

Now

$$C_{4,2} = \begin{pmatrix} 0_2 & 0_2 & 0_2 & -J_2 \\ 0_2 & 0_2 & J_2 & 0_2 \\ 0_2 & -J_2 & 0_2 & 0_2 \\ J_2 & 0_2 & 0_2 & 0_2 \end{pmatrix} \quad (22)$$

From the definition of the map $\Theta_{C, I}$ we see that

$$C_{4,2} = \Theta_{C, I} \left[\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \right] = \Theta_{C, I} [iM_{1 \otimes k}] \quad (23)$$

Remark 4.1 In other words, the matrix $\check{J}_4 = M_{1 \otimes k}$ once again *intervenes in the form of Clifford conjugation* for $Cl(4, 2)$, as it did for $Cl(3, 2)$ - a fact that would be difficult to arrive at without the $\mathbf{H} \otimes \mathbf{H}$ basis for $M(4, \mathbf{R})$!

Since $p + q > 5$, the spin group of $Cl(4, 2)$ is the subset of those elements in $\{Y \in M(8, \mathbf{R}), YG_{4,2} = G_{4,2}Y, C_{4,2}^T Y^T C_{4,2} Y = I_8\}$ which satisfy the further restriction that they leave the space of 1-vectors invariant. To identify that subset we work at the level of the Lie algebras, cf., Remark (2.3). Specifically, dimension considerations show that the Lie algebra $spin^+(4, 2)$ must be a 15-dimensional Lie subalgebra of

$$\{Y \in M(8, \mathbf{R}), YG_{4,2} = G_{4,2}Y, C_{4,2}^T Y^T = -YC_{4,2}\}, \quad (24)$$

which leaves the space of 1-vectors invariant under the action $Z \rightarrow ZY - YZ$, where Y belongs to the set in Equation (24), and Z is a 1-vector. Equivalently it suffices to find an element in the set in Equation (24) which violates this condition. The obvious candidate for this element is $\Theta_{C,I}(iI_4)$. To that end let $\{Y_1, \dots, Y_6\}$ be the basis in Equation (21) and check if even one $\text{Tr}\{Y_k^T[\Theta_{C,I}(iI_4)Y_i - Y_i\Theta_{C,I}(iI_4)]\}$, for some fixed $i \in \{1, \dots, 6\}$ and each $k = 1, \dots, 6$ is zero. Using the fact that $\Theta_{C,I}(iI_4)$ is the *triple Kronecker product* $I_2 \otimes I_2 \otimes i\sigma_y$ and that the matrices Y_i are *themselves triple Kronecker products* it is seen, in fact, that these traces vanish for all $i = 1, \dots, 6$. This shows that the Lie algebra of the spin group must be the $\Theta_{C,I}$ images of matrices in the following set:

$$\{Z \in M(4, \mathbf{C}) : (iM_{1 \otimes k})^* Z^* = -Z(iM_{1 \otimes k}); \text{Tr}(Z) = 0\}$$

Let us next relate these calculations to $SU(2, 2)$ being isomorphic to $Spin^+(4, 2)$. To that end the main observation is that the Hermitian matrix, $iM_{1 \otimes k}$, is unitarily conjugate to $I_{2,2}$. This unitary conjugation can, in fact, be explicitly found and this relies on the fact that $M_{1 \otimes k}$ also equals the *Kronecker product*, $\sigma_x \otimes (i\sigma_y)$. Explicit orthonormal eigenpairs for each of the 2×2 matrices σ_x and $i\sigma_y$ are easily found. They are

- σ_x : $\{(1, u_1), (-1, u_2)\}$ where $u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- $i\sigma_y$: $\{(i, v_1), (-i, v_2)\}$, where $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Therefore from elementary properties of Kronecker products, it follows that an explicit orthonormal eigenpairs for $C_{4,2}$ are given by

$$\{(1, u_1 \otimes v_2), (1, u_2 \otimes v_1), (-1, u_1 \otimes v_1), (-1, u_2 \otimes v_2)\}$$

Therefore $U^* C_{4,2} U = I_{2,2}$, where

$$U = [u_1 \otimes v_2 \mid u_2 \otimes v_1 \mid u_1 \otimes v_1 \mid u_2 \otimes v_2]$$

$$U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -i & i & i & -i \\ 1 & -1 & 1 & -1 \\ -i & -i & i & i \end{pmatrix} \quad (25)$$

These calculations can now be summarized as

Theorem 4.2 $Cl(4, 2) = M(8, R)$ has the matrices Y_i given by Equation (21) as a basis of 1-vectors. With respect to this basis

- The grade involution is $Y \rightarrow \tilde{J}_8^T Y \tilde{J}_8$.
- Clifford conjugation is given by $Y \rightarrow [\Theta_{C,I}(iM_{1 \otimes k})]^T Y^T [\Theta_{C,I}(iM_{1 \otimes k})]$.
- $Spin^+(4, 2)$ equals those matrices in $Y \in M(8, R)$ satisfying the conditions: i) Y is a $\Theta_{C,I}$ matrix; ii) $\det(Y) = 1$; iii) $[\Theta_{C,I}(iM_{1 \otimes k})]^T Y^T [\Theta_{C,I}(iM_{1 \otimes k})] Y = I_8$.

- $spin^+(4, 2)$ consists of those matrices $Y \in M(8, \mathbf{R})$ satisfying the following conditions: i) Y is a $\Theta_{\mathbf{C}, \mathbf{I}}(Z)$ matrix with $Z \in M(4, \mathbf{C})$ matrix; ii) $\text{Tr}(Z) = 0$; iii) $[\Theta_{\mathbf{C}, \mathbf{I}}(iM_{1 \otimes k})]^T Y^T = -Y[\Theta_{\mathbf{C}, \mathbf{I}}(iM_{1 \otimes k})]$.

The group of matrices in $M(4, \mathbf{C})$ whose $\Theta_{\mathbf{C}, \mathbf{I}}$ images equal $Spin^+(4, 2)$ is explicitly conjugate to $SU(2, 2)$ via the matrix U in Equation (25).

5 Other Low Dimensional $Spin^+(p, q)$

In this appendix our approach to $Spin^+(p, q)$, when $(p, q) \in \{(2, 1), (2, 2), (3, 1), (4, 1), (1, 5), (3, 3)\}$ is outlined. Of particular note is the completely straightforward derivation of the covering $SL(2, \mathbf{C}) \rightarrow SO^+(3, 1, \mathbf{R})$ and the somewhat surprising concentric embedding of $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ in $SL(4, \mathbf{R})$ as $Spin^+(2, 2)$.

5.1 $Spin^+(2, 1)$

A basis of 1-vectors for $Cl(2, 1)$ is given by Equation (1). Quite clearly $Cl(2, 1)$ is the algebra of 2×2 block matrices with each block a 2×2 real diagonal matrix.

Reversion and Clifford conjugation with respect to this basis are given by Equations (2) and (3) respectively. So, the grade involution is $G_{2,1}(X) = (J_4 M_{1 \otimes k})^T X (J_4 M_{1 \otimes k}) = M_{1 \otimes i}^T X M_{1 \otimes i}$.

Hence, writing an $X \in Cl(2, 1)$ in the form $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with each of A, B, C, D 2×2 real diagonal, it is seen that X is even iff A and D commute with J_2 and B and C anti-commute with J_2 . So $A = a\sigma_z; B = bI_2; C = cI_2; D = d\sigma_z$. Such an even X belongs to $Spin^+(2, 1)$ iff it also belongs to $Sp(4, \mathbf{R}) = \{X \in M(4, \mathbf{R}) : J_4^T X^T J_4 X = I_4\}$. But, this is equivalent to $B^T D$ and $A^T C$ symmetric and $A^T D - C^T D = I_2$. Since A, B, C, D are all diagonal the first two conditions are superfluous. Similarly, the diagonality of A, \dots, D , shows that $A^T D - C^T D = (ad - bc)I_2$. So we conclude that an even $X \in Cl(2, 1)$ is in $Spin^+(2, 1)$ iff $ad - bc = 1$.

Thus $Spin^+(2, 1)$ is isomorphic to $SL(2, \mathbf{R})$. Specifically, $Spin^+(2, 1)$ is the group of 4×4 real matrices obtained by embedding $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$, in the form

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ c & 0 & d & 0 \\ 0 & c & 0 & -d \end{pmatrix}$$

5.2 $Spin^+(3, 1)$

The most natural basis of 1-vectors for $Cl(2, 0)$

$$\{\sigma_z, \sigma_x\}$$

With respect to this basis reversion on $Cl(2, 0)$ is $X \rightarrow X^T$, while Clifford conjugation is $X \rightarrow J_2^T X^T J_2$.

Application of \mathbf{IC} then produces the following:

1. A basis of 1-vectors for $Cl(3, 1) = M(4, \mathbf{R})$ given by

$$\{\sigma_z \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_x \otimes I_2, i\sigma_y \otimes I_2\}$$

2. With respect to this last basis reversion on $Cl(3, 1)$ is $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow K_4^T \begin{pmatrix} J_2^T A^T J_2 & J_2^T C^T J_2 \\ J_2^T B^T J_2 & J_2^T D^T J_2 \end{pmatrix} K_4 = M_{1 \otimes k}^T X^T M_{1 \otimes k}$

3. Clifford conjugation is $X \rightarrow J_4^T X^T J_4$

4. The grade automorphism is therefore $X \rightarrow (-M_{1 \otimes i})^T X (-M_{1 \otimes i})$

Since $Cl(3, 1)$ is $M(4, \mathbf{R})$ we would like the grade automorphism to be given by J_4 instead of $-M_{1 \otimes i}$. To that end, we need a unit quaternion q such that $\bar{q}(-i)q = j$. One choice is $q = \frac{1}{\sqrt{2}}(1 + k)$. Now $J_4 = M_{1 \otimes q}^T (-M_{1 \otimes i}) M_{1 \otimes q}$. Accordingly we change the basis of 1-vectors for $Cl(3, 1)$ to $\{Y_i\}$, where $Y_i = M_{1 \otimes q}^T X_i M_{1 \otimes q}$, with the X_i the basis of 1-vectors above.

Then invoking Remark (2.4), it is seen that:

1. With respect to this basis the grade involution is precisely $X \rightarrow J_4^T X J_4$.

2. Clifford conjugation is, with respect to this basis of 1-vectors precisely $X \rightarrow M_{1 \otimes i}^T X^T M_{1 \otimes i}$.

Thus,

$$Spin^+(3, 1) = \{Y \in M(4, \mathbf{R}) : Y J_4 = J_4 Y; M_{1 \otimes i}^T Y^T M_{1 \otimes i} Y = I_4\} \quad (26)$$

Remark (2.10) yields that the first condition is equivalent to $Y = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$, with $A, B \in M(2, \mathbf{R})$.

The second condition says

$$\begin{pmatrix} -J_2 A^T J_2 A + J_2 B^T J_2 B & J_2 A^T J_2 B + J_2 B^T J_2 A \\ -J_2 B^T J_2 A - J_2 A^T J_2 B & J_2 B^T J_2 B - J_2 A^T J_2 A \end{pmatrix} = I_4$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Then saying $Y \in Spin^+(3, 1)$ is equivalent to the following four conditions:

- (1, 1) block = I_2 , i.e., $-J_2 A^T J_2 A + J_2 B^T J_2 B = I_2$. A quick calculation shows that this is equivalent to $ad - bc + \beta\gamma - \alpha\delta = 1$.
- (1, 2) block = 0_2 , i.e., $J_2 A^T J_2 B + J_2 B^T J_2 A = 0_2$. Equivalently, $b\gamma - \alpha d + \beta c - a\delta = 0$
- (2, 1) block = 0_2 . But the (2, 1) block, $-J_2 B^T J_2 A - J_2 A^T J_2 B$ is precisely minus the (1, 2) block. So this condition is now redundant.
- (2, 2) block = I_2 . But the (2, 2) block, $J_2 B^T J_2 B - J_2 A^T J_2 A$, is identical to the (1, 1) block.

Now consider $\det(A + iB) = (a + i\alpha)(d + i\delta) - (b + i\beta)(c + i\gamma) = (ad - \alpha\delta + \beta\gamma - bc) + i(ad + \alpha d - b\gamma - \beta c) = +i0 = 1$. Thus, indeed $Spin^+(3, 1)$ is the $\Theta_{\mathbf{C}, \mathbf{II}}$ image of $SL(2, \mathbf{C})$ in $M(4, \mathbf{R})$, and hence isomorphic as a group to $SL(2, \mathbf{C})$.

5.3 $Spin^+(2, 2)$

Observing first that a basis of 1-vectors for $Cl(1, 1)$ is $\{\sigma_x, i\sigma_y\}$, with respect to which reversion and Clifford conjugation are given respectively by $X \rightarrow \sigma_x^T X^T \sigma_x$ and $X \rightarrow J_2^T X^T J_2$, and then applying **IC** to this data yields the following:

- A basis of 1-vectors $\{\sigma_z \otimes \sigma_x, \sigma_x \otimes I_2, \sigma_z \otimes i\sigma_y, i\sigma_y \otimes I_2\}$ for $Cl(2, 2)$
- Reversion $Cl(2, 2)$, with respect to this basis, is $X \rightarrow K_4^T \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}^T X^T \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} K_4$, i.e., it is $X \rightarrow M_{1 \otimes k}^T X^T M_{1 \otimes k}$.
- Clifford conjugation on $Cl(2, 2)$, with respect to this basis, is given by $X \rightarrow J_4^T \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}^T X^T \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} J_4 = M_{-k \otimes 1}^T X^T M_{-k \otimes 1}$

- Hence the grade automorphism on $Cl(2, 2)$, with respect to this basis of 1-vectors, is $X \rightarrow M_{k \otimes k}^T X M_{k \otimes k}$

Now $M_{k \otimes k}$ is precisely $\begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix} = \sigma_z \otimes \sigma_z$. So $X \in Cl(2, 2) = M(4, \mathbf{R})$ is even its blocks $X_{ij}, i, j = 1, 2$ satisfy

$$\begin{pmatrix} X_{11}\sigma_z & -X_{12}\sigma_z \\ X_{21}\sigma_z & -X_{22}\sigma_z \end{pmatrix} = \begin{pmatrix} \sigma_z X_{11} & \sigma_z X_{12} \\ -\sigma_z X_{21} & -\sigma_z X_{22} \end{pmatrix}$$

So X_{11}, X_{22} commute with σ_z and X_{12}, X_{21} anticommute with σ_z . Equivalently, X_{11}, X_{22} are both diagonal and X_{12}, X_{21} are both anti-diagonal.

Thus, an even vector X in $Cl(2, 2) = M(4, \mathbf{R})$ is given by

$$X = \begin{pmatrix} a & 0 & 0 & b \\ 0 & d & c & 0 \\ 0 & \beta & \alpha & 0 \\ \gamma & 0 & 0 & \delta \end{pmatrix}$$

Such an even X is in $Spin^+(2, 2)$ iff it satisfies $X^{cc}X = I_4$. Equivalently

$$\begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} \begin{pmatrix} X_{11}^T & X_{21}^T \\ X_{12}^T & X_{22}^T \end{pmatrix} \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = I_4$$

This, in turn is equivalent to the following 4 conditions:

- (1, 1) block = I_2 . This is the same as requiring $\begin{pmatrix} a\delta - b\gamma & 0 \\ 0 & \alpha\delta - c\beta \end{pmatrix} = I_2$
- (1, 2) block = 0_2 . But the (1, 2) block is equals $\begin{pmatrix} 0 & b\delta - c\beta \\ c\alpha - \beta\gamma & 0 \end{pmatrix}$. So this is no constraint.
- Similarly, (2, 1) block = 0_2 is no constraint either.
- (2, 2) block = I_2 . The (2, 2) block is $\sigma_x X_{11}^T \sigma_x X_{22} - \sigma_x X_{21}^T \sigma_x X_{12}$, which equals $\begin{pmatrix} \alpha\delta - \beta\gamma & 0 \\ 0 & a\delta - b\gamma \end{pmatrix}$.

But this is precisely the first constraint again.

Hence we can conclude that $X = \begin{pmatrix} a & 0 & 0 & b \\ 0 & d & c & 0 \\ 0 & \beta & \alpha & 0 \\ \gamma & 0 & 0 & \delta \end{pmatrix}$ is in $Spin^+(2, 2)$ iff the matrices $\begin{pmatrix} a & b \\ \gamma & \delta \end{pmatrix}$ and

$\begin{pmatrix} d & c \\ \beta & \alpha \end{pmatrix}$ are both in $SL(2, R)$. Thus $Spin^+(2, 2)$ is isomorphic to $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ embedded concentrically in $M(4, \mathbf{R})$.

5.4 $Spin^+(3, 3)$

Applying **IC** to the basis of 1 vectors for $Cl(2, 2)$ in the previous subsection leads to the following initial basis of 1-vectors for $Cl(3, 3)$:

$$\{\sigma_z \otimes \sigma_z \otimes \sigma_x, \sigma_z \otimes \sigma_x \otimes I_2, \sigma_x \otimes I_2 \otimes I_2, \sigma_z \otimes \sigma_z \otimes i\sigma_y, \sigma_z \otimes i\sigma_y \otimes I_2, i\sigma_y \otimes I_2 \otimes I_2\} \quad (27)$$

A calculation plus the observation that $\begin{pmatrix} 0 & M_{1 \otimes k} \\ -M_{1 \otimes k} & 0 \end{pmatrix} = i\sigma_y \otimes \sigma_x \otimes i\sigma_y$, shows that with respect to this basis Clifford conjugation on $Cl(3, 3)$ is

$$X \rightarrow (i\sigma_y \otimes \sigma_x \otimes i\sigma_y)^T X^T (i\sigma_y \otimes \sigma_x \otimes i\sigma_y)$$

Similarly, using the observation that $M_{k \otimes 1} = i\sigma_y \otimes \sigma_x$, one finds that reversion, with respect to this basis, is given by

$$X \rightarrow (\sigma_x \otimes i\sigma_y \otimes \sigma_x)^T X^T (\sigma_x \otimes i\sigma_y \otimes \sigma_x)$$

Hence, using basic properties of the Kronecker product, it is seen that the grade involution, with respect to this basis, is

$$X \rightarrow (\sigma_z \otimes \sigma_z \otimes \sigma_z)^T X (\sigma_z \otimes \sigma_z \otimes \sigma_z) = \sigma_z \otimes \sigma_z \otimes \sigma_z X \sigma_z \otimes \sigma_z \otimes \sigma_z$$

Unlike the case of $Spin^+(2, 2)$ the block structure of the even vectors in $Cl(3, 3)$ with respect to this choice of the grade morphism is no longer very informative. So we look for an explicit conjugation to a more palatable version of the grade morphism. Now $\sigma_z \otimes \sigma_z \otimes \sigma_z$ is diagonal with four 1s and four -1 s on the diagonal. Hence, by inspection it is explicitly permutation symmetric to $diag(I_4, -I_4)$. Specifically, if

$$P = [e_1 \mid e_4 \mid e_6 \mid e_7 \mid e_2 \mid e_3 \mid e_5 \mid e_8]$$

Then $P^T(\sigma_z \otimes \sigma_z \otimes \sigma_z)P = diag(I_4, -I_4)$.

We would like the grade involution to be given $X \rightarrow diag(I_4, -I_4)Xdiag(I_4, -I_4)$, so that an even vector is then precisely the set of all block-diagonal 8×8 matrices [cf., Remark (2.15)]. To that end we change basis of 1 vectors according to

$$X_i = P^T Y_i P, i = 1, \dots, 6$$

where the Y_i form the basis in Equation (27).

Then a calculation provides the basis $\{X_i\}, i = 1, \dots, 6$ via

$$\{\sigma_x \otimes \sigma_z \otimes I_2, \sigma_x \otimes \sigma_z \otimes \sigma_x, \sigma_x \otimes \sigma_x \otimes I_2, i\sigma_y \otimes I_2 \otimes I_2, \sigma_x \otimes \sigma_z \otimes i\sigma_y, \sigma_x \otimes i\sigma_y \otimes I_2\} \quad (28)$$

Then Clifford conjugation, with respect to the basis X_i , is given by $X \rightarrow C_{3,3}^T X^T C_{3,3}$, where we have

$$C_{3,3} = P^T (i\sigma_x \otimes \sigma_x \otimes i\sigma_y) P$$

The matrix $C_{3,3}$ is given a neat form by availing of the observation that $i\sigma_x \otimes \sigma_x \otimes i\sigma_y = [e_8 \mid -e_7 \mid e_6 \mid -e_5 \mid -e_4 \mid e_3 \mid -e_2 \mid e_1]$.

Hence,

$$C_{3,3} = \begin{pmatrix} 0_4 & M_{1 \otimes k} \\ -M_{1 \otimes k} & 0_4 \end{pmatrix} \quad (29)$$

So $Spin^+(3, 3)$ is a subgroup of block-diagonal 8×8 matrices $\begin{pmatrix} A & 0_4 \\ 0_4 & D \end{pmatrix}$ satisfying two additional conditions:

- $C_{3,3}^T \begin{pmatrix} A & 0_4 \\ 0_4 & D \end{pmatrix}^T C_{3,3} \begin{pmatrix} A & 0_4 \\ 0_4 & D \end{pmatrix} = I_8$.
- $\begin{pmatrix} A & 0_4 \\ 0_4 & D \end{pmatrix} X \begin{pmatrix} A & 0_4 \\ 0_4 & D \end{pmatrix}^{CC}$ is a 1-vector for all 1-vectors X .

The first of these conditions is equivalent to the condition that $-M_{1 \otimes k} A^T M_{1 \otimes k} D = I_4$. So $A \in GL(4, \mathbf{R})$ can be taken to be arbitrary and then D is prescribed as $D = -M_{1 \otimes k} A^{-T} M_{1 \otimes k}$.

Remark 5.1 It is useful to write down the linearization of the conditions in the last paragraph, because that describes the Lie algebra of $Spin^+(3, 3)$. The linearization of these conditions describes a block-diagonal $\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$, with $R, S \in M(4, \mathbf{R})$ satisfying

$$\begin{pmatrix} R^T & 0_4 \\ 0_4 & S^T \end{pmatrix} \begin{pmatrix} 0_4 & M_{1 \otimes k} \\ -M_{1 \otimes k} & 0_4 \end{pmatrix} = - \begin{pmatrix} 0_4 & M_{1 \otimes k} \\ -M_{1 \otimes k} & 0_4 \end{pmatrix} \begin{pmatrix} R & 0_4 \\ 0_4 & S \end{pmatrix}$$

This is equivalent to $R^T M_{1 \otimes k} = -M_{1 \otimes k} S$ and $-S^T M_{1 \otimes k} = M_{1 \otimes k} R$. Once again these two conditions are equivalent to one another and thus to $S = M_{1 \otimes k} R^T M_{1 \otimes k}$. Thus $spin^+(3, 3)$ is the collection of block-diagonal matrices $Z = \begin{pmatrix} R & 0 \\ 0 & M_{1 \otimes k} R^T M_{1 \otimes k} \end{pmatrix}$ satisfying the additional condition that $ZX - XZ$ is a 1-vector for all 1-vectors.

Let us use this to show that $Spin^+(3, 3)$ is indeed $SL(4, \mathbf{R})$. For this, using Remark (2.3), it is easier to work with the linearized version of the condition $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} X \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^{CC}$ is a 1-vector for all 1-vectors, and show that it is violated by the prototypical element of $M(4, \mathbf{R})$ which is not in $sl(4, \mathbf{R})$ - viz., I_4 or rather its embedding as an element of $M(8, \mathbf{R})$ satisfying the conditions in Remark (5.1). Since $M_{1 \otimes k} I_4 M_{1 \otimes k} = -I_4$ we have to show that $Z = \begin{pmatrix} I_4 & 0_4 \\ 0_4 & -I_4 \end{pmatrix} = \sigma_z \otimes I_2 \otimes I_2$ violates the condition $ZX - XZ$ is a 1-vector for all 1-vectors X . This is easily verified by using elementary properties of the Kronecker product - especially **KP3**.

Hence we conclude that $spin^+(3, 3) = \left\{ \begin{pmatrix} R & 0 \\ 0 & M_{1 \otimes k} R^T M_{1 \otimes k} \end{pmatrix}, R \in sl(4, \mathbf{R}) \right\}$. Hence, $Spin^+(3, 3)$ is a group of 8×8 real matrices of the form $\left\{ \begin{pmatrix} A & 0 \\ 0 & -M_{1 \otimes k} A^{-T} M_{1 \otimes k} \end{pmatrix}, A \in SL(4, \mathbf{R}) \right\}$. This group is evidently isomorphic to $SL(4, \mathbf{R})$.

5.5 $Spin^+(4, 1)$

Consider the following basis of 1-vectors for $Cl(3, 0) = M(2, \mathbf{C})$:

$$\{\sigma_x, \sigma_y, \sigma_z\}$$

Reversion with respect to this basis of 1-vectors is given by $X \rightarrow X^*$ and Clifford conjugation is $X \rightarrow J_2^T X^T J_2$. Therefore **IC** applied to this data yields the following:

- A basis of 1-vectors for $Cl(4, 1) = M(4, \mathbf{C})$:

$$\{Y_1 = \sigma_z \otimes \sigma_x, Y_2 = \sigma_z \otimes \sigma_y, Y_3 = \sigma_z \otimes \sigma_z, Y_5 = K_4 = \sigma_x \otimes I_2, Y_5 = J_4 = i\sigma_y \otimes I_2\}$$

- Clifford conjugation with respect to this basis is $X \rightarrow J_4^T X^* J_4$
- Reversion with respect to this basis is $X \rightarrow M_{1 \otimes k}^T X^T M_{1 \otimes k}$.
- Hence the grade involution, with respect to this basis of 1-vectors, is given by $X \rightarrow (J_4 M_{1 \otimes k})^T \bar{X} (J_4 M_{1 \otimes k})$. Since $J_4 = M_{1 \otimes j}$, the grade involution is $X \rightarrow M_{1 \otimes i}^T \bar{X} M_{1 \otimes i}$.

In view of $Cl(4, 1)$ being the algebra of an even sized (i.e., 4) complex matrices, it is prudent to seek a basis of 1-vectors with respect to which the grade involution is given by $X \rightarrow J_4^T \bar{X} J_4$. Now $J_4 = M_{1 \otimes j} = M_{1 \otimes q}^T M_{1 \otimes i} M_{1 \otimes q}$, for a unit quaternion q satisfying $\bar{q} i q = j$. For instance, pick $q = \frac{1}{\sqrt{2}}(1 - k)$. Accordingly we change the basis of 1-vectors for $Cl(4, 1)$ to $X_i = M_{1 \otimes q}^T Y_i M_{1 \otimes q}$.

Then with respect to this basis of 1-vectors:

- Grade involution is $X \rightarrow M_{1 \otimes j}^T \bar{X} M_{1 \otimes j}$.
- Clifford conjugation is given by $X \rightarrow (M_{1 \otimes q}^T J_4 M_{1 \otimes q})^T X^* (M_{1 \otimes q}^T J_4 M_{1 \otimes q})$. But $M_{1 \otimes q}^T J_4 M_{1 \otimes q} = M_{1 \otimes -i}$ as can be seen quickly from quaternionic algebra. So Clifford conjugation with respect to this basis is $X \rightarrow M_{1 \otimes i}^T X^* M_{1 \otimes i}$.

Thus, even vectors precisely those matrices in $M(4, \mathbf{C})$ which are in the image of Θ_H and thus we have

Proposition 5.2 $Spin^+(4, 1) = \{X \in M(4, \mathbf{C}) : X \in Im(\Theta_H); M_{1 \otimes i}^T X^* M_{1 \otimes i} X = I_4\}$.

5.6 $Spin^+(1, 5)$

Let us start with the following basis of 1-vectors for $Cl(0, 4) = M(2, \mathbf{H})$:

$$\begin{aligned} Z_1 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ Z_2 &= \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \\ Z_3 &= \begin{pmatrix} 0 & k \\ i & 0 \end{pmatrix} \\ Z_4 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

With respect to this basis of 1- vectors, Z_i , some calculations reveal that : i) Clifford Conjugation is given by $X \rightarrow X^*$; ii) Reversion is given by $X \rightarrow \sigma_z X^* \sigma_z$; and hence iii) the grade involution is expressible as $X \rightarrow \sigma_z X \sigma_z$.

Applying **IC** to this basis yields a basis $\{Y_i\}, i = 1, \dots, 6$ of 1-vectors for $Cl(1, 5)$, given by

$$\sigma_x \otimes I_2, \sigma_z \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \sigma_z \otimes \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \sigma_z \otimes \begin{pmatrix} 0 & k \\ i & 0 \end{pmatrix}, \sigma_z \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i\sigma_y \otimes I_2$$

IC then also shows that, with respect to the basis of 1-vectors Y_i :

- Clifford conjugation is given by $X \rightarrow J_4^* \begin{pmatrix} \sigma_z^* & 0 \\ 0 & \sigma_z^* \end{pmatrix} X^* \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} J_4 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}^* X^* \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}$
- Reversion is given by $X \rightarrow K_4^* X^* K_4$
- Hence the grade involution is $X \rightarrow [\begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} K_4]^* X \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} K_4 = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix} X \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}$

Once again by Remark (2.15), it would be preferable to have the grade involution described by $X \rightarrow \text{diag}(I_2, -I_2) X \text{diag}(I_2, -I_2)$, so that even vectors become block-diagonal matrices. This is achieved by mere inspection, since $P^T \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix} P = \text{diag}(I_2, -I_2)$, where P is permutation matrix

$$P = [e_1 \mid e_4 \mid e_2 \mid e_3]$$

Accordingly we change the basis of 1-vectors to X_i , with $X_i = P^T Y_i P$. We then find, after some block matrix manipulations that

- $X_1 = P^T Y_1 P = \sigma_x \otimes \sigma_x$
- $X_2 = P^T Y_2 P = \sigma_x \otimes i\sigma_z$
- $X_3 = P^T Y_3 P = i\sigma_y \otimes I_2 = J_4$
- $Y_4 = P^T X_4 P = \sigma_x \otimes j\sigma_z$. Here the j is the quaternion unit j .
- $Y_5 = P^T X_5 P = \sigma_x \otimes k\sigma_z$. Similarly, the k is the quaternion unit k .
- $X_6 = P^T Y_6 P = \sigma_x \otimes i\sigma_y$

Clifford conjugation, with respect to this basis of 1-vectors, is given by $X \rightarrow C_{1,5}^* X^* C_{1,5}$, where

$$C_{1,5} = P^T \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = i\sigma_y \otimes \sigma_x.$$

Thus $Spin^+(1, 5)$ consists of those $X \in M(4, \mathbf{H})$ which satisfy : i) X is block-diagonal $diag(X_1, X_2)$, with each $X_i \in M(2, \mathbf{H})$, ii) $X^{cc}X = I_4$, and iii) $X^{cc}ZX$ is a 1-vector for all 1-vectors Z .

Focussing on the condition, $X^{cc}X = I_4$, it is found that it is equivalent to $\sigma_x X_2^* \sigma_x X_1 = I_2$ and $\sigma_x X_1^* \sigma_x X_2 = I_2$. But the first of these two conditions is equivalent to the second, as follows by $*$ -conjugating the first equation and then pre and post-multiplying the result by σ_x . Finally, the first of these conditions is equivalent to X_1 being invertible and $X_2^{-1} = \sigma_x X_1^* \sigma_x$; and iii) $X^{cc}ZX$ is a 1-vector for all 1-vectors Z .

Thus, $Spin^+(1, 5)$ is isomorphic to the collection of matrices in $M(4, \mathbf{H})$ satisfying:

1. X is block-diagonal with diagonal blocks X_1, X_2 both invertible in $M(2, \mathbf{H})$, and $X_2^{-1} = \sigma_x X_1^* \sigma_x$, and
2. $X^{cc}ZX$ is a 1-vector for all 1-vectors Z .

To identify the restriction imposed by the last condition it is easier to work with the Lie algebra $spin^+(1, 5)$. This should be some subalgebra of block-diagonal matrices $diag(A_1, A_2)$, with $A_i \in M(2, H)$ satisfying $A_2 = \sigma_x A_1^* \sigma_x$. To specify this subalgebra, we use by way of variation the second alternative in Remark (2.3), i.e., the space of bivectors (with respect to the basis of 1-vectors X_i) is explicitly computed, since this equals $spin^+(1, 5)$.

This computation yields the following as the basis of the space of bivectors:

1. $X_1 X_2 = I_2 \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $X_1 X_3 = I_2 \otimes \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$; $X_1 X_4 = I_2 \otimes \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$; $X_1 X_5 = -\sigma_x \otimes \sigma_x$; $X_1 X_6 = I_2 \otimes -\sigma_z$.
2. $X_2 X_3 = I_2 \otimes k I_2$; $X_2 X_4 = -I_2 \otimes j I_2$; $X_2 X_5 = -\sigma_z \otimes i \sigma_z$; $X_2 X_6 = I_2 \otimes i \sigma_x$.
3. $X_3 X_4 = I_2 \otimes i I_2$; $X_3 X_5 = -\sigma_z \otimes j \sigma_z$; $X_3 X_6 = I_2 \otimes j \sigma_x$.
4. $X_4 X_5 = -\sigma_z \otimes k \sigma_z$; $X_4 X_6 = I_2 \otimes k \sigma_x$.
5. $X_5 X_6 = \sigma_z \otimes i \sigma_y$.

Thus,

- The typical element in the span of $\{X_1 X_2, X_1 X_3, X_1 X_4\}$ is given by $I_2 \otimes \begin{pmatrix} 0 & p \\ \bar{p} & 0 \end{pmatrix}$, with $p \in \mathbf{P}$.
- The typical element in the span of $\{X_2 X_5, X_3 X_5, X_4 X_5\}$ is given by $-\sigma_z \otimes \begin{pmatrix} q & 0 \\ 0 & \bar{q} \end{pmatrix}$, with $q \in \mathbf{P}$.
- The typical element in the span of $\{X_2 X_6, X_3 X_6, X_4 X_6\}$ is given by $I_2 \otimes \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$, with $s \in \mathbf{P}$.
- The typical element in the span of $\{X_1 X_6, X_2 X_3, X_2 X_4, X_3 X_4\}$ is given by $I_2 \otimes \begin{pmatrix} \alpha & 0 \\ 0 & -\bar{\alpha} \end{pmatrix}$, with $\alpha \in \mathbf{H}$.
- The typical element in the span of $\{X_1 X_5, X_5 X_6\}$, is given by $\sigma_z \otimes \begin{pmatrix} 0 & r_1 \\ r_2 & 0 \end{pmatrix}$, with $r_1, r_2 \in R$.

Thus, a typical bivector is a block-diagonal matrix, with each block in $M(2, \mathbf{H})$. Furthermore, the NW block of a bivector, which also determines the SE block, is of the form:

$$\begin{pmatrix} \alpha - q & p + s + r_1 \\ \bar{p} + s + r_2 & -\bar{q} - \bar{\alpha} \end{pmatrix}$$

This is clearly of the form

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

with $q_{ij} \in \mathbf{H}$ arbitrary, except for $\text{Re}(q_{11} + q_{22}) = 0$. Thus, in view of Remark (2.14), the NW block, X_{11} , can be chosen to be any element of $sl(2, \mathbf{H})$, while the SE block, X_{22} is prescribed by $X_{22} = -\sigma_x X_{11}^* \sigma_x$. This confirms that $spin^+(1, 5)$ is indeed isomorphic to $sl(2, \mathbf{H})$ whence $Spin^+(1, 5)$ is isomorphic to $SL(2, \mathbf{H})$.

6 Conclusions

In this work we have developed what we consider to be a straightforward approach to the indefinite spin groups $Spin^+(p, q)$ for $p + q \leq 6$. By working in the matrix algebra that $Cl(p, q)$ is isomorphic to, as opposed to the matrix algebra that its even subalgebra is isomorphic to, one can now address applications requiring the spin group to live in the same matrix algebra as the one-vectors (thereby eliminating ad hoc constructions). Furthermore the method has a flavour which is closer to the defining conditions for the spin groups making it, in our opinion, didactically more palatable. As byproducts we find novel representations for some of the classical groups, such as the Λ embedding of $\check{Sp}(4, \mathbf{R})$ in $M(8, \mathbf{R})$ and the concentric embedding of $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ in $SL(4, \mathbf{R})$, and more natural interpretations of some of the members of the $\mathbf{H} \otimes \mathbf{H}$ basis of $M(4, \mathbf{R})$. Some obvious questions arise out of this work:

- Though $Spin^+(q, p)$ is isomorphic to $Spin^+(p, q)$, the same is not true for $Cl(q, p)$ and $Cl(p, q)$. Thus, it would be a profitable exercise to apply this approach to study $Spin^+(q, p)$ corresponding to the pairs (p, q) studied in this work. It is reasonable to expect this to lead to more novel representations for some of the classical groups.
- A natural question is how much further this method extends to higher dimensions. The form of the iteration **IC** in conjunction with basic block matrix algebra renders the extension of the benefits of the quaternion tensor product a distinct possibility in higher dimensions as well (indeed, even in this work most of the $Cl(p, q)$ s were isomorphic to algebras of matrices of size already larger than 4×4). Thus finding bases of 1-vectors, matrix forms of Clifford conjugation etc., ought to be possible. The real “bottleneck” is the analysis of the third condition in the definition of the spin groups [See IV) of Definition (2.1)]. The methods outlined in Remark (2.3) would have to be substantially extended to this end.
- It would be worthwhile to complete the analysis started in this paper to compute exactly and explicitly the exponentials of matrices in Lie algebras other than $spin(3, 2, \mathbf{R})$. Of course, some of the other $so(p, q, \mathbf{R})$ s, such $so(2, \mathbf{R})$ etc., are even simpler, since now one has to exponentiate only 2×2 matrices. But the explicit computation of the minimal polynomials of matrices in $su(2, 2)$, $SL(4, \mathbf{R})$ etc., - a worthwhile task with applications going beyond exponentiation - is needed for the other $so(p, q, \mathbf{R})$ s and will require significant effort.

7 Appendix - Spin Groups and Matrix Exponentiation

Computing the exponential of a matrix is central to much of applied mathematics. In general, this is quite arduous, [14]. However, for matrices $X \in so(p, q, \mathbf{R})$, the theory of Clifford Algebras and spin groups enables the reduction of finding e^X , with $X \in so(p, q, \mathbf{R})$, to the computation of e^Y , where Y is the associated element in the Lie algebra $spin^+(p, q)$, via the method detailed below. Typically Y is smaller in size than X . In particular, the minimal polynomial of Y is typically of lower degree than that of X and often is also more structured. The method is described in the following algorithm, which is an adaptation of a similar algorithm in [8]:

Algorithm 7.1 Step 1 Identify a collection of matrices which serve as a basis of 1 vectors for the Clifford Algebra $Cl(p, q)$.

Step 2 Identify the explicit form of Clifford conjugation (ϕ^{cc}) and the grade (or so-called main) automorphism on $Cl(p, q)$ with respect to this collection of matrices. Equivalently identify the explicit form of Clifford conjugation and reversion (ϕ^{rev}) with respect to this collection of matrices.

Step 3 Steps 1 and 2 help in identifying both the spin group $Spin^+(p, q)$ and its Lie algebra $spin^+(p, q)$ as sets of matrices, within the same matrix algebra, that the matrices in Step 1 live in. Hence, the double covering $\Phi_{p,q} : Spin^+(p, q) \rightarrow SO^+(p, q, \mathbf{R})$ the matrix, with respect to the basis of 1-vectors in Step 1, of the linear map $H \rightarrow ZH\phi^{cc}(Z)$, with H a matrix in the collection of 1-vectors in Step 1 and $Z \in Spin^+(p, q)$. This enables one to express $\Phi_{p,q}(Z)$ as a matrix in $SO^+(p, q, \mathbf{R})$.

Step 4 Linearize $\Phi_{p,q}$ to obtain the Lie algebra isomorphism $\Psi_{p,q} : spin^+(p, q) \rightarrow so(p, q, \mathbf{R})$. This reads as $W \rightarrow YW - WY$, with W once again a 1-vector and $Y \in spin^+(p, q)$. Once again this leads to a matrix in $so(p, q, \mathbf{R})$, which is $\Psi_{p,q}(Y)$.

Step 5 Given $X \in so(p, q, \mathbf{R})$, find $\Psi_{p,q}^{-1}(X) = Y \in spin^+(p, q)$.

Step 6 Compute the matrix e^Y and use Step 3 to find the matrix $\Phi_{p,q}(e^Y)$. This matrix is e^X .

With the contents of this work, this algorithm will be executable for $X \in so(p, q, \mathbf{R})$, for those pairs (p, q) considered in this work, as long as there are closed form formulae for finding the e^Y of Step 6 above. For each such pair, (p, q) , considered in this work, the corresponding Y is atmost 4×4 . So the minimal polynomials of such Y is atmost a quartic. For some pairs (such as $(p, q) = (3, 2)$, for instance) the list of possible minimal polynomials has even greater structure, [18], and thus the attendant problem of exponentiation becomes even simpler. Obtaining the minimal polynomials of such Y 's is left to future work.

References

- [1] R. Ablamowicz, "Matrix Exponential Via Clifford Algebras" *J. Nonlinear Mathematical Physics*, **5**, 294-313, 1998.
- [2] Y. Ansari & V. Ramakrishna, "On The Non-compact Portion of $Sp(4, \mathbf{R})$ Via Quaternions", *J. Phys A: Math. Theor*, **41**, 335203, 1-12, (2008).
- [3] G. Chen, D. Church, B. Englert, C. Henkel, B. Rohnwedder, M. Scully & M. Zubairy, *Quantum Computing Devices: Principles, Design and Analysis*, Chapman & Hall CRC Press, Boca Raton, (2006).
- [4] U. Fano & A. R. P. Rau, *Symmetries in Quantum Physics*, Academic Press, San Diego (1996).
- [5] H. Fassbender, D. Mackey & N. Mackey, Hamilton and Jacobi Come Full Circle: Jacobi Algorithms For Structured Hamiltonian Eigenproblems", *Linear Algebra & its Applications*, **332**, 37- 80, (2001).
- [6] R. Gilmore, *Lie Groups, Physics and Geometry*, Cambridge University Press, Cambridge (2008).
- [7] D. Hacon, "Jacobi's Method for Skew-Symmetric Matrices", *SIAM J. Matrix Analysis*, **14**, 619 - 628, (1993).
- [8] E. Herzig, V. Ramakrishna & M. Dabkowski, "Note on Reversion, Rotation and Exponentiation in Dimensions Five and Six", *J. Geometry and Symmetry in Physics*, **35**, doi 10.7546, 61-101, (2014).
- [9] R. A. Horn & C. R. Johnson, *Matrix Analysis*, Cambridge University Press (1990).

- [10] C. R. Johnson, T. Laffé & C. K. Li, “Linear Transformations on $M_n(\mathbf{R})$ That Preserve the Ky Fan k -Norm and a Remarkable Special Case When $(n, k) = (4, 2)$, *Linear and Multilinear Algebra*, **23**, 285 - 298, (1988).
- [11] P. Lounesto, *Clifford Algebras and Spinors*, II edition, Cambridge University Press (2002).
- [12] N. Mackey, “Hamilton and Jacobi Meet Again - Quaternions and the Eigenvalue Problem”, *Siam J. Matrix Analysis*, **16**, 421 - 435, (1995).
- [13] D. Mackey, N. Mackey & S. Dunleavy, “ Structure Preserving Algorithms for Perplectic Eigenproblems”, *Electronic Journal of Linear Algebra*, **13**, 10 - 39, (2005).
- [14] C. Moler and C. Van Loan, “Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty - Five Years Later”, *SIAM Review*, **45**, No.1, 3 -19 (2003).
- [15] I. R. Porteous, *Clifford Algebras and the Classical Groups*, Cambridge U Press, (2009).
- [16] V. Ramakrishna & F. Costa, “On the Exponential of Some Structured Matrices”, *J. Phys. A - Math & General*, **Vol 37**, 11613-11627, 2004.
- [17] V. Ramakrishna & H. Zhou, “On the Exponential of Matrices in $\mathfrak{su}(4)$ ”, *J. Phys. A - Math & General*, **39** (2006), 3021-3034.
- [18] V. Ramakrishna, Y. Ansari & F. Costa, ‘Minimal Polynomials of Some Matrices Via Quaternions’, *Advances in Applied Clifford Algebras*, **22**, pg 159-183, (2012).